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DIRECTIONAL DIFFERENTIABILITY OF METRIC PROJECTION IN $H_0^2(\Omega)$ AND APPLICATIONS

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Directional differentiability of metric projection in $\mathbf{H}_0^2(\boldsymbol{\Omega})$ and applications

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Abstract

The form of directional derivative of the metric projection in the Sobolev space $H_0^2(\Omega)$ onto the convex set

$$K = \{f \in H_0^2(\Omega) \mid f \geq \psi\}$$

is derived.

The result is used to obtain the differential stability of solutions to obstacle problems for the Kirchhoff plate. Applications to the shape design sensitivity of the obstacle problems are provided.

Key words: metric projection, differential stability, variational inequality, obstacle problem, tangent cone, shape derivative, Kirchhoff plate.

AMS(MOS) subject classification : 49B22, 49A29, 49A22, 93B30

Dérivée directionnelle d'une projection dans $\mathbf{H}_0^2(\boldsymbol{\Omega})$ et applications

Résumé

La forme de la dérivée directionnelle de la projection dans l'espace de Sobolev $H_0^2(\Omega)$ sur le convexe

$$K = \{f \in H_0^2(\Omega) \mid f \geq \psi\}$$

est obtenue.

Le résultat est utilisé pour établir la stabilité par rapport au domaine des solutions du problèmes d'obstacle dans le cas de la plaque de Kirchhoff. On présente des applications au dessin optimal.

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1. INTRODUCTION

The paper is concerned with the differential stability of metric projection in the Sobolev space $H_0^2(\Omega)$ onto a translation of the cone of nonnegative elements.

We provide the new results on the directional differentiability of the metric projection in the Sobolev space $H_0^2(\Omega)$ onto the convex set $K = \{f \in H_0^2(\Omega) \mid f \geq \psi\}$, where $\Omega \subset \mathbb{R}^d$ is an open, bounded domain.

In particular, we derive the form of tangent cone $T_K(f)$ for any element $f \in K$ - see Theorem 1. The same argument can be used for the convex set

$$K = \{f \in H_0^m(\Omega) \mid f \geq \psi\}, \quad m = 2, 3, ...$$

where $\psi \in H^m(\Omega)$, $\psi < 0$ on $\partial\Omega$.

The differential stability of metric projection in the Sobolev space $H_0^1(\Omega)$ onto the cone of nonnegative elements is considered by Mignot in [15]. Mignot derived the form of the so-called conical differential of the metric projection. However, the technique of proof used by Mignot is based on potential theory in Dirichlet spaces, therefore, his argument cannot be directly applied in the Sobolev space $H_0^2(\Omega)$.

In section 3 we provide the sufficient conditions under which set K is polyhedric [10],[15], at a given point $f \in K$. The question of polyhedricity is adressed here since it implies directional differentiability of the metric projection onto K with the explicit form of the differential [10],[15]. We refer the reader to [10], [15] for related results in the Sobolev space $H_0^1(\Omega)$.

The applications of our main results provided in Section 4 are interesting on their own. In section 4.1 the form of the shape derivative for solutions of an obstacle problem is obtained. In section 4.2 the necessary optimality conditions for an optimal design problem for the Kirchhoff plate are derived. In section 5 the case of simply supported Kirchhoff plate in the Sobolev space $H^2(\Omega) \cap H_0^1(\Omega)$ is considered.

We refer the reader to [10],[15] for the related results on the differential stability of metric projections in Hilbert space. Some applications of the differential stability of metric projection onto convex sets in Sobolev spaces are presented in [2]-[5], [11],[18]-[24]. In particular the sensitivity analysis of solutions of constrained optimization problems is studied in [20],[21],[22]. The applications to optimal design problems are presented in [2],[3]-[5],[11],[23],[24]. We refer the reader to [6],[8] for general results on variational inequalities, and to [1],[7],[13],[26] for the results on the Sobolev spaces.

We recall some properties of the Sobolev spaces and the notion of capacity [26]. The Sobolev spaces $H_0^1(\Omega)$ and $H_0^2(\Omega)$ are the closures of $C_0^\infty(\Omega)$ with norms

$$\|\varphi\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla\varphi|^2 dx$$

$$\|\varphi\|^2_{H_0^2(\Omega)} = \int_\Omega |\Delta\varphi|^2 dx$$

respectively. If $\varphi \in H_0^2(\Omega)$, from the definition $\frac{\partial\varphi}{\partial x_i} \in H_0^1(\Omega)$ for each $i = 1, 2, ..., d$. Functions in $H_0^1(\Omega)$ are defined quasi everywhere and are quasi continuous. These notions are made precise below.

The C_1−capacity of a compact set F is defined as

$$C_1(F) = \inf\{\int |\nabla\varphi|^2 dx \ : \ \varphi \geq 1 \text{ on } F, \ \ 0 \leq \varphi \in C_0^\infty(\Omega)\}$$

similarly C_2−capacity

$$C_2(F) = \inf\{\int |\Delta\varphi|^2 dx \ : \ \varphi \geq 1 \text{ on } F, \ \ 0 \leq \varphi \in C_0^\infty(\Omega)\} \ .$$

The capacity of a Borel set is then defined as the supremum of capacities of its compact subsets. A statement holds C_i−q.e., $i = 1, 2$, if it holds except for a set of C_i−capacity zero. With this definition we have the following results:

1. Let $\varphi \in H_0^1(\Omega)$, and $\{\varphi_n\} \subset C_0^\infty(\Omega)$ converge to φ in $H_0^1(\Omega)$. Then a subsequence of $\{\varphi_n\}$ converge C_1- q.e. and this is a representative of φ.
2. Let $\varphi \in H_0^1(\Omega)$. Then φ has a quasicontinuous representative: There is a representative $\overline{\varphi}$ such that given $\varepsilon > 0$, there is an open set $U(\varepsilon)$ of C_1−capacity less than ε such that the restriction of $\overline{\varphi}$ to the complement of $U(\varepsilon)$ is continuous.
3. Any two quasi continuous representatives of $\varphi \in H_0^1(\Omega)$ agree C_1−q.e.
4. Every set of positive Lebesgue measure has positive C_1−capacity.

The same holds for the Sobolev space $H_0^2(\Omega)$ with respect to C_2−capacity [26]. We use standard notation throughout the paper [1],[26].

2. TANGENT CONE

We shall consider the metric projection onto the following convex set

$$K = \{f \in H_0^2(\Omega) \mid f(x) \geq \psi(x), x \in \Omega\} \tag{2.1}$$

with respect to the scalar product

$$(y, z) = \int_\Omega \Delta y(x) \Delta z(x) dx \ . \tag{2.2}$$

We assume that $\psi \in H^2(\Omega)$, $\psi(x) < 0$ on $\partial\Omega$ C_2−q.e., therefore set (2.1) is nonempty. The metric projection $z = P_K y$, $y \in H_0^2(\Omega)$, is given by the unique solution of the following variational inequality

$$z \in K \ : \ \int_\Omega \Delta z(x)\Delta(\varphi - z)(x)dx \geq \int_\Omega \Delta y(x)\Delta(\varphi - z)(x)dx, \ \ \forall\varphi \in K \ . \tag{2.3}$$

We denote

$$C_K(z) = \{\varphi \in H_0^2(\Omega) \mid \exists t > 0 \text{ such that } z + t\varphi \in K\} \ . \tag{2.4}$$

We derive the form of tangent cone $T_K(z) = \text{cl}(C_K(z))$, cl stands for closure, for any element z in convex set (2.1).

THEOREM 1

For any element $z \in K$, the tangent cone $T_K(z)$ takes the form

$$T_K(z) = \{\varphi \in H_0^2(\Omega) \mid \varphi(x) \geq 0 \ , \ C_2 - \text{q.e. on } \Xi\} \tag{2.5}$$

where $\Xi = \{x \in \Omega \mid z(x) = \psi(x)\} \subset \Omega$.

PROOF

Note that $C_K(z)$ and hence also $T_K(z)$ is a convex cone containing all non-negative elements of $H_0^2(\Omega)$. Let an element $V \in H_0^2(\Omega)$ be given and suppose that $V \geq 0$ C_2-q.e. on Ξ. There exists the unique element $\phi_0 \in T_K(z)$ such that

$$\|V - \phi_0\|^2_{H_0^2(\Omega)} = \inf \{\|V - \phi\|^2_{H_0^2(\Omega)} \mid \phi \in C_K(z)\} \ . \tag{2.6}$$

It is easy to see that for any $H_0^2(\Omega) \ni \phi \geq 0$, $t \geq 0$, $\phi_0 + t\phi \in T_K(z)$. Using (2.6) and standard arguments it follows

$$(V - \phi_0, \phi)_{H_0^2(\Omega)} \leq 0 \ , \ 0 \leq \phi \in H_0^2(\Omega) \tag{2.7}$$

hence there exists a non-negative Radon measure μ on Ω such that

$$(V - \phi_0, \phi)_{H_0^2(\Omega)} = -\int \phi d\mu \ , \ \phi \in C_0^\infty(\Omega) \ . \tag{2.8}$$

This implies in particular that for $\phi \geq 0$

$$\int \phi d\mu = -(V - \phi_0, \phi)_{H_0^2(\Omega)} \leq \|V - \phi_0\|_{H_0^2(\Omega)} \|\phi\|_{H_0^2(\Omega)} \ .$$

So by definition of C_2-capacity we see μ cannot charge sets of zero C_2-capacity. Since the measure may be large near the boundary it is not clear that (2.8) holds for all $\phi \in H_0^2(\Omega)$. We can circumvent this difficulty by repeated use of a result of L. I. Hedberg: Theorem 3.1 in [13]. First we show that (2.8) holds for any bounded $\phi \in H_0^2(\Omega)$ which is non-negative and has compact support. Indeed for suitable mollifiers ϱ_n , $\phi \star \varrho_n \in C_0^\infty(\Omega)$, have compact support, and tend boundedly pointwise C_2-q.e. and in $H_0^2(\Omega)$ to ϕ. Since μ is Radon measure which does not charge the $0-$capacity sets we may appeal to Lebesgue dominated convergence to finish the claim. In the general case if $0 \leq \phi \in H_0^2(\Omega)$ by the above theorem of Hedberg , we can select $0 \leq w_k \leq 1$, $k = 1, 2, ...$ such that $w_k\phi$ has compact support and is in L^∞ approximating ϕ in $H_0^2(\Omega)$. In particular $w_k\phi$ converges to ϕ C_2-q.e. By (2.8) we have

$$\int w_k \phi d\mu = -(V - \phi_0, w_k\phi)_{H_0^2(\Omega)}$$

is bounded, so by Fatou Lemma $\phi \in L^1(\mu)$. On the other hand $w_k\phi \leq \phi$ so dominated convergence applies

$$-\int \phi d\mu = (V - \phi_0, \phi)_{H_0^2(\Omega)} \ , \ 0 \leq \phi \in H_0^2(\Omega) \ . \tag{2.9}$$

Now let $\phi \in C_0^\infty(\Omega)$, $0 \le \phi \le 1$, then $\phi(z-\psi) \in H_0^2(\Omega)$. We show that

$$\phi_0 + t\phi(z-\psi) \in T_K(z), \quad -1 \le t \le 1 \ .$$

It is sufficient to show that for any $\varphi \in C_K(z)$, it follows $\varphi + t\phi(z-\psi) \in C_K(z)$. Now $\varepsilon\varphi + z - \psi \ge 0$ in Ω for some $\varepsilon > 0$, hence for $s > 0$, $\frac{s}{1-s} < \varepsilon$ we have

$$s[\varphi + t\phi(z-\psi)] + z - \psi \ge 0, \text{ in } \Omega$$

since $(1+st\phi)(z-\psi) \ge (1-s)(z-\psi)$.

Using this in (2.6) with ϕ replaced by $\phi_0 + t\phi(z-\psi)$ we obtain

$$(V-\phi_0, \phi(z-\psi))_{H_0^2(\Omega)} = 0$$

which, because $\phi(z-\psi)$ has compact support and belongs to $H_0^2(\Omega)$ means

$$\int \phi(z-\psi)d\mu = 0$$

hence

$$\mu(x \ : \ z > \psi) = 0$$

i.e. μ is concentrated on Ξ. Our next step is to show that $\phi_0 = 0$ μ-a.e. To this end using the fact that $T_K(z)$ is a cone and taking $t\phi_0$ for ϕ in (2.6) we get

$$(V-\phi_0, \phi_0)_{H_0^2(\Omega)} = 0 \ . \tag{2.10}$$

Now we use Hedberg's result once more. Choose w_k , $0 \le w_k \le 1$ such that $w_k\phi_0$ has compact support and converges to ϕ_0 in $H_0^2(\Omega)$. Since $\phi_0 \ge 0$ on Ξ and μ is concentrated on Ξ, $w_k\phi_0 \le \phi_0$ μ-a.e. So using the same argument as above we get

$$0 = (V-\phi_0, \phi_0)_{H_0^2(\Omega)} = -\int \phi_0 d\mu$$

i.e. that $\phi_0 = 0$ μ-a.e.

Finally since $\phi_0 = 0$ μ-a.e and $V \ge 0$ C_2-q.e. on Ξ we can repeat the above argument to get

$$(V-\phi_0, V-\phi_0)_{H_0^2(\Omega)} = -\int (V-\phi_0)d\mu = -\int V d\mu \ .$$

But the right hand side is ≤ 0 because $V \ge 0$ C_2-q.e. on Ξ and therefore μ-a.e., thus $V = \phi_0$.

REMARK 1

For $d = 1,2,3$ proof of Theorem 1 simplifies since by the Sobolev embedding theorem $H_0^2(\Omega) \subset C(\overline{\Omega})$. It is clear that

$$T_K(z) \subset \{\varphi \in H_0^2(Q) \mid \varphi(x) \geq 0\ , \quad \text{on } \Xi\}$$

therefore it is sufficient to show that any element $V(.) \geq 0$ on Ξ actually belongs to $T_K(z)$.

Ξ is compact, hence there exists $0 \leq \eta \in C_0^\infty(\Omega)$, $\eta \equiv 1$ on Ξ. Since by the Sobolev embedding theorem $z, \psi, V \in C(\overline{\Omega})$ therefore for any $\varepsilon > 0$ there exists $t > 0$ such that

$$t(V + \varepsilon\eta) + z - \psi \geq 0, \quad \text{in } \Omega\ .$$

Thus

$$V + \varepsilon\eta \in C_K(z), \quad \forall\varepsilon > 0$$

and

$$V + \varepsilon\eta \to V \quad \text{in } H_0^2(Q) \quad \text{strongly with } \varepsilon \downarrow 0$$

hence $V \in \text{cl}(C_K(z)) = T_K(z)$.

3. DIFFERENTIABILITY OF METRIC PROJECTION

We derive new result on the differentiability of metric projection P_K in the Hilbert space $H = H_0^2(\Omega)$ onto convex closed set $K \subset H$ of the form (2.1). Let $T_K(f)$ be the tangent cone to K at $f \in K$. $T_K(f)$ is the closure in the space $H_0^2(\Omega)$ of the convex cone

$$C_K(f) = \{v \in H_0^2(\Omega) \mid \exists t > 0 \text{ such that } f(x) + tv(x) \geq \psi(x) \text{ in } \Omega\}\ . \tag{3.1}$$

For a given element $g \in H_0^2(\Omega)$, such that $f = P_K(g)$ let us define the following convex cone in the space $H_0^2(\Omega)$

$$S = T_K(f) \cap [g - P_K(g)]^\perp = T_K(f) \cap [f - g]^\perp\ . \tag{3.2}$$

DEFINITION 1

The set $K \subset H_0^2(\Omega)$ is polyhedric at $f \in K$, if for any $g \in H_0^2(\Omega)$ such that $f = P_K g$ it follows

$$T_K(f) \cap [f - g]^\perp = \text{cl}(C_K(f) \cap [f - g]^\perp)\ . \tag{3.3}$$

REMARK 2

Let us recall [10],[15] that if condition (3.3) is satisfied for given elements $(f, g) \in H_0^2(\Omega) \times H_0^2(\Omega)$, $f = P_K(g)$ then for all $h \in H_0^2(\Omega)$ and for $t > 0$, t small enough

$$P_K(g + th) = P_K g + tP_S h + o(t)\ . \tag{3.4}$$

In such a case the metric projection P_K is conically differentiable, in the notation of [15], at $g \in H_0^2(\Omega)$. It turns out that condition (3.3) is satisfied if the support of the non-negative Radon measure defined below by (3.5) is admissible in the following way.

DEFINITION 2

A compact set F is admissible if for any element $\varphi \in H_0^2(\Omega)$, $\varphi = 0$ C_2-q.e. on F implies $\varphi \in H_0^2(\Omega \setminus F)$.

We denote by $B(x,r)$, $x \in \mathbb{R}^d$, $r > 0$, the ball of radius r and center x, $|A|$ denotes the Lebesgue measure of any set $A \subset \mathbb{R}^d$.

PROPOSITION 1

Let $F \subset \Omega$ be compact and assume that the following holds :
for C_1-quasi every $x \in F$ and for all $r > 0$, r small enough,

$$|F \cap B(x,r)| > 0 \ .$$

Then F is admissible.

PROOF

By Theorem 1.1 in [13] it is sufficient to show the following: let $\varphi \in H_0^2(\Omega)$ and $\varphi = 0$ C_2-q.e. on F. Then $\nabla\varphi = 0$ C_1-q.e. on F. Now $\varphi \in H_0^1(\Omega)$ so by a standart result, $\nabla\varphi = 0$ a.e. on F. Since $\varphi \in H_0^2(\Omega)$, each component of $\nabla\varphi$ belongs to $H_0^1(\Omega)$ and hence has a finely continuous version [26]. If for $x \in F$, $|\nabla\varphi|(x) > 0$ then in a fine neighborhood of x the same inequality will be obtained. Since finely open sets have positive measure, and since $\nabla\varphi = 0$ a.e. on F, this violates our condition on F. Thus $\nabla\varphi = 0$ C_1-q.e. on F.

For any $f \in H_0^2(\Omega)$, $g = P_K f$, denote by $\nu \geq 0$ the Radon measure defined as follows

$$-\int \varphi d\nu = \int_\Omega \Delta(g-f)\Delta\varphi dx, \ \forall\varphi \in C_0^\infty(\Omega) \tag{3.5}$$

and by $\mathcal{S}_0$ the following convex cone

$$\mathcal{S}_0 = \{\varphi \in H_0^2(\Omega \setminus F) \mid \ \varphi \geq 0 \ \ C_2\text{-q.e. on } \Xi\}, \tag{3.6}$$

where $F = F_f = \mathrm{spt}\nu \subset \Xi$ is compact, $\mathrm{spt}\nu$ denotes the support of Radon measure ν.

THEOREM 2

For any $f \in H_0^2(\Omega)$ with admissible compact set F_f it follows that

$$\mathrm{cl}(C_K(f) \cap [f-g]^\perp) = \mathcal{S}_0 \ . \tag{3.7}$$

PROOF

It is clear that

$$\mathrm{cl}(C_K(f) \cap [f-g]^\perp) \subset \mathcal{S}_0 \ .$$

Therefore we show that

$$\mathcal{S}_0 \subset \operatorname{cl}(C_K(f) \cap [f-g]^\perp) .$$

For the convenience of the reader we repeat here the argument already used in the proof of Theorem 1.

$C_K(f) \cap [f-g]^\perp$ and hence also $\operatorname{cl}(C_K(f) \cap [f-g]^\perp)$ is a convex cone containing all non-negative elements of $H_0^2(\Omega \setminus F)$. Let an element $V \in \mathcal{S}_0$ be given, therefore in view of (3.7) $V \in H_0^2(\Omega \setminus F)$ and $V \geq 0$ on Ξ. Suppose that $V \notin \operatorname{cl}(C_K(f) \cap [f-g]^\perp)$, then there exists a unique element $\phi_0 \in \operatorname{cl}(C_K(f) \cap [f-g]^\perp)$ such that

$$\|V-\phi_0\|^2_{H_0^2(\Omega\setminus F)} = \inf \{\|V-\phi\|^2_{H_0^2(\Omega\setminus F)} \mid \phi \in C_K(f) \cap [f-g]^\perp\} \tag{3.8}$$

i.e. ϕ_0 is the metric projection of V onto $\operatorname{cl}(C_K(f) \cap [f-g]^\perp)$.

Using (3.8) it follows

$$(V-\phi_0, \phi)_{H_0^2(\Omega\setminus F)} \leq 0 \, , \; 0 \leq \phi \in H_0^2(\Omega \setminus F)$$

hence there exists a non-negative Radon measure μ on $\Omega \setminus F$ such that

$$(V-\phi_0, \phi)_{H_0^2(\Omega\setminus F)} = -\int \phi d\mu \, , \; \phi \in C_0^\infty(\Omega \setminus F) . \tag{3.9}$$

This implies in particular that for $\phi \geq 0$

$$\int \phi d\mu = -(V-\phi_0, \phi)_{H_0^2(\Omega\setminus F)} \leq \|V-\phi_0\|_{H_0^2(\Omega\setminus F)} \|\phi\|_{H_0^2(\Omega\setminus F)} \cdot$$

So μ cannot charge sets of zero capacity. In order to show that (3.9) holds for all $\phi \in H_0^2(\Omega \setminus F)$, first we show that (3.9) holds for any $\phi \in H_0^2(\Omega \setminus F)$ which is non-negative and has compact support. For suitable mollifiers ϱ_n , $\phi \star \varrho_n \in C_0^\infty(\Omega \setminus F)$, have compact support, tend in $C(\Omega \setminus F)$ and $H_0^2(\Omega \setminus F)$ to ϕ. Since μ is Radon measure it follows by Lebesgue dominated convergence that

$$\lim_{n\to\infty} \int \phi \star \varrho_n d\mu = \int \phi d\mu .$$

In the general case if $0 \leq \phi \in H_0^2(\Omega \setminus F)$, we can select $0 \leq w_k \leq 1$, $k = 1, 2, \ldots$ such that $w_k\phi$ has compact support and is in $C(\Omega \setminus F)$ approximating ϕ in $H_0^2(\Omega \setminus F)$. By (3.9) we have

$$\int w_k \phi d\mu = -(V-\phi_0, w_k\phi)_{H_0^2(\Omega\setminus F)} \tag{3.10}$$

is bounded, so by Fatou Lemma $\phi \in L^1(\mu)$. On the other hand $w_k\phi \leq \phi$ so dominated convergence applies

$$-\int \phi d\mu = (V-\phi_0, \phi)_{H_0^2(\Omega\setminus F)} \, , \; 0 \leq \phi \in H_0^2(\Omega \setminus F) . \tag{3.11}$$

Now let $\varphi \in C_0^\infty(\Omega \setminus \Xi)$, $0 \le \varphi \le 1$, $-1 \le t \le 1$, then $\phi_0 + t\varphi \in$ $\mathrm{cl}(C_K(f) \cap [f-g]^\perp)$. Using this in (3.8) with $\phi = \phi_0 + t\varphi$ we obtain

$$(V - \phi_0, \varphi)_{H_0^2(\Omega \setminus F)} = 0$$

which means

$$\int \varphi d\mu = 0$$

hence

$$\mu(x \notin \Xi) = 0$$

i.e. μ is concentrated on Ξ. Our next step is to show that $\phi_0 = 0$ μ-a.e. To this end using the fact that $cl(C_K(f) \cap [f-g]_\perp)$ is a cone and taking $t\phi_0$ for ϕ in (3.8) we get

$$(V - \phi_0, \phi_0)_{H_0^2(\Omega \setminus F)} = 0\ . \tag{3.12}$$

So using the same argument as in the proof of Theorem 1 we get

$$0 = (V - \phi_0, \phi_0)_{H_0^2(\Omega \setminus F)} = -\int \phi_0 d\mu$$

i.e. that $\phi_0 = 0$ μ-a.e.

Finally since $\phi_0 = 0$ μ-a.e and $V \ge 0$ on Ξ it follows that

$$(V - \phi_0, V - \phi_0)_{H_0^2(\Omega \setminus F)} = -\int (V - \phi_0) d\mu = -\int V d\mu\ .$$

But the right hand side is ≤ 0 because $V \ge 0$, thus $V = \phi_0$.

PROPOSITION 2

Let $f \in K \subset H_0^2(\Omega)$ be given, $\Xi = \{x \in \Omega \mid f(x) = \psi(x)\}$.

For any Radon measure $\nu \in H^{-2}(\Omega)$, $\nu \ge 0$, $\mathrm{spt}\nu \subset \Xi$, there exists $g \in H_0^2(\Omega)$, $f = P_K g$, such that

$$\int_\Omega \Delta(f-g)\Delta\varphi dx = \int \varphi d\nu, \quad \forall \varphi \in H_0^2(\Omega) \tag{3.13}$$

PROOF

Let $g \in H_0^2(\Omega)$ satisfy

$$\int_\Omega \Delta g \Delta\varphi dx = \int_\Omega \Delta f \Delta\varphi dx - \int \varphi d\nu, \quad \forall \varphi \in H_0^2(\Omega)$$

We have $f = P_K g$. To see it let us observe that

$$\int \varphi d\nu \ge 0, \quad \forall \varphi \in T_K(f) \tag{3.14}$$

since $\eta - f \in T_K(f),\ \forall \eta \in K$ it follows

$$\int (\eta - f) d\nu \geq 0, \quad \forall \eta \in K \tag{3.15}$$

hence

$$\int (\eta - f) d\nu = \int_\Omega \Delta(f - g)\Delta(\eta - f) dx \geq 0, \quad \forall \eta \in K \tag{3.16}$$

which shows that $f = P_K g$.

COROLLARY 1

Assume that $F = \mathrm{spt}\nu$ is admissible then (3.3) and (3.4) hold, where cone S is defined by (3.6).

Therefore, condition (3.3) can be satisfied if $C_1(\Xi) = 0$. The set K is polyhedric at f iff the set $\Xi = \{x | f(x) = \psi(x)\}$ has capacity zero.

4. APPLICATIONS TO OPTIMAL DESIGN

We apply our differentiability results to optimization problems for the fourth order elliptic variational inequalities. The variational inequality for the Kirchhoff model of an elastic plate with an obstacle is considered, however the results can be used as well as for the von Karman plate. In section 4.1 we derive the form of shape derivative for the solution of the obstacle problem. In section 4.2 the first order necessary optimality conditions for an optimum design problem are obtained.

4.1. SHAPE SENSITIVITY ANALYSIS OF OBSTACLE PROBLEMS FOR KIRCHHOFF PLATES

Let $\Omega \in \mathbb{R}^2$ be a given domain with smooth boundary $\Gamma = \partial\Omega$. We derive the form of the so-called shape (Lagrange) derivative of the solution of an obstacle problem for the Kirchhoff plate in the direction of a vector field $V(.,.)$. The proof of this result uses the material derivative method developed in Sokołowski and Zolesio [24] for the shape sensitivity analysis of variational inequalities combined with Theorem 1.

Let there be a given vector field

$$V(.,.) \in C^1(0,\delta; C^2(\mathbb{R}^2;\mathbb{R}^2)) \tag{4.1}$$

where $\delta > 0$ is a given constant. We define a family $\{\Omega_t\} \subset \mathbb{R}^2, t \in [0,\delta)$, of domains as follows [24]:

$$\begin{aligned}\Omega_t &= T_t(V)(\Omega) \\ &= \{x \in \mathbb{R}^2 \mid \exists X \in \Omega \text{ such that } x(0) = X, x = x(t)\}\end{aligned} \tag{4.2}$$

where $x(t) \in \mathbb{R}^2,\ t \in [0,\delta)$ is given by the unique solution of the following system

$$\begin{aligned}&\frac{dx}{dt} = V(t, x(t)), \quad t \in (0,\delta) \\ &x(0) = X\ .\end{aligned} \tag{4.3}$$

Let us assume that $\psi(.) \in H^3(\mathbb{R}^n)$ is a given element such that $\psi(x) \leq 0$ in some open neighborhood in $\mathbb{R}^2$ of the manifold $\partial\Omega \subset \mathbb{R}^2$. Thus for $t > 0, t$ small enough the set

$$K(\Omega_t) = \{\varphi \in H_0^2(\Omega) \mid \varphi(x) \geq \psi(x), \text{ in } \Omega_t\} \tag{4.4}$$

is a non-empty, closed convex subset of the Sobolew space $H_0^2(\Omega)$. We use the same symbol $\psi(.)$ to denote the restriction of $\psi(.) \in H^2(\mathbb{R}^2)$ to the domain $\Omega_t, t \in [0, \delta)$. Let there be given an element $f \in H^2(\mathbb{R}^2)$. We denote by $w_t \in H^2(\Omega)$, $t \in [0, \delta)$, the unique solution of the following variational inequality

$$\begin{aligned} &w_t \in K(\Omega_t) \ : \\ &\int_{\Omega_t} \Delta w_t \Delta(\varphi - w_t) dx \geq \int_{\Omega_t} f(\varphi - w_t) dx, \quad \forall \varphi \in K(\Omega_t) \ . \end{aligned} \tag{4.5}$$

For $t = 0$, we denote $w(x) = w_0(x)$, $x \in \Omega$, and let $\nu \geq 0$ be the Radon measure defined by

$$\int \varphi d\nu = \int_\Omega \{\Delta w \Delta \varphi - \varphi f\} dx \ .$$

Furthermore we denote by $\tilde{w}_t(x)$, $x \in \mathbb{R}^2$, an extension of the element $w_t(x)$, $x \in \Omega$, $t \in [0, \delta)$ defined as follows

$$\tilde{w}_t(x) = \begin{cases} w_t(x), & x \in \Omega_t, \ t \in [0, \delta) \\ 0, & x \in \mathbb{R}^2 \setminus \Omega_t, \ t \in [0, \delta) \end{cases} \tag{4.6}$$

It can be shown using the material derivative method [24] combined with Theorem 1 that the element $\tilde{w}_{t|\Omega} \in H^2(\Omega)$ is right-differentiable with respect to t, at $t = 0$. We denote $V(0) = V(0, .)$.

THEOREM 3

Assume that sptν is admissible in the sense of Definition 2. Let $\nabla w.V(0) \in H^2(\Omega)$.

Then for $t > 0, t$ small enough

$$\tilde{w}_{t|\Omega} = w + tw' + o(t), \quad \text{in } H^2(\Omega) \tag{4.7}$$

where $\|o(t)\|_{H^2(\Omega)}/t \to 0$ with $t \downarrow 0$.

The shape derivative $w' \in H^2(\Omega)$ is given by a unique solution of the following variational inequality

$$\begin{aligned} w' \in S_v = &\{\varphi \in H^2(\Omega) \cap H_0^1(\Omega) \mid \frac{\partial \varphi}{\partial n} = -v \frac{\partial^2 w}{\partial n^2} \text{ on } \partial\Omega, \\ &\varphi(x) \geq 0 \text{ on } \Xi^+, \quad \varphi(x) = 0 \text{ on } \Xi^0\} \end{aligned} \tag{4.8}$$

$$\int_\Omega \Delta w' \Delta(\varphi - w') dx \geq 0, \quad \forall \varphi \in S_v(\Omega) \tag{4.9}$$

here we denote $v(x) = \langle V(0,x), n(x)\rangle_{\mathbb{R}^2}$, $x \in \partial\Omega$; $n(x)$, $x \in \partial\Omega$ is the unit outward normal vector on $\partial\Omega$ and $\Xi^0 = \text{spt}\nu$,

$$\Xi^+ = \Xi \setminus \Xi^0 \tag{4.10}$$

$$\Xi = \{x \subset \Omega \mid w(x) = \psi(x)\} . \tag{4.11}$$

PROOF

Let us recall that the shape derivative $w' = w'(\Omega)$ of the solution $w = w(\Omega)$ to variational inequality

$$w \in K(\Omega): \quad \int_\Omega \Delta w \Delta(\varphi - w)dx \geq \int_\Omega f(\varphi - w)dx, \quad \forall \varphi \in K(\Omega) \tag{4.12}$$

in the direction of a vector field $V(.,.)$ is defined as follows :

$$w' = \dot{w} - \nabla w.V(0) \tag{4.13}$$

where

$$\dot{w} = \lim_{t\downarrow 0}(w_t \circ T_t - w)/t \tag{4.14}$$

here $w_t \in H_0^2(\Omega)$ denotes the unique solution of variational inequality (4.5). First we derive the form of material derivative $\dot{w}$. To this end we transport variational inequality (4.5) to the fixed domain Ω using the mapping $T_t \ : \ X \to x = x(t)$ defined by (4.3). It follows that the element :

$$w^t = w \circ T_t \in H_0^2(\Omega), \ t \in [0, \delta) \tag{4.15}$$

satisfies :

$$\begin{gathered} w^t \in K^t = \{\varphi \in H_0^2(\Omega) \mid \varphi \geq \psi^t \text{ in } \Omega\} \\ Q^t(w^t, \varphi - w^t) \geq \int_\Omega f^t(\varphi - w^t)dx, \quad \forall \varphi \in K^t \end{gathered} \tag{4.16}$$

here we denote

$$\psi^t = \psi \circ T_t, \quad f^t = \gamma_t f \circ T_t, \quad \gamma_t = \det(DT_t),$$

DT_t denotes the Jacobian of the mapping $T_t : \ \mathbb{R}^2 \to \mathbb{R}^2$

$$Q^t(z, \varphi) = \int_\Omega \gamma_t^{-1} \text{div}(A_t.\nabla z)\text{div}(A_t.\nabla\varphi)dx, \quad \forall z, \varphi \in H_0^2(\Omega) \tag{4.17}$$

$$\text{with } A_t = \gamma_t DT_t^{-1}.(DT_0^{-1})^* . \tag{4.18}$$

Let us denote

$$z^t = w^t - \psi^t, \quad \xi^t(\varphi) = \text{div}(A_t.\nabla\varphi) \tag{4.19}$$

hence

$$z^t \in K_0 = \{\varphi \in H_0^2(\Omega) \mid \varphi \geq 0 \text{ in } \Omega\}$$
$$Q^t(z^t, \varphi - z^t) \geq \int_\Omega f^t(\varphi - z^t)dx - Q^t(\psi^t, \varphi - z^t), \ \forall \varphi \in K_0 \ . \tag{4.20}$$

By application of Theorem 1 combined with the abstract result derived in [23] (Theorem 1, p.1419) it follows that for $t > 0$, t small enough :

$$z^t = z^0 + t\dot{z} + o(t), \quad \text{in } H_0^2(\Omega) \tag{4.21}$$

where $\dot{z} \in H_0^2(\Omega)$ satisfies the following variational inequality

$$\dot{z} \in S(\Omega) = \{\varphi \in H_0^2(\Omega) \mid \varphi \geq 0 \text{ on } \Xi^+, \ \varphi = 0 \text{ on } \Xi^0\} \ : \tag{4.22}$$
$$\int_\Omega \Delta\dot{z}\Delta(\varphi - \dot{z})dx \geq \int_\Omega \dot{f}(\varphi - \dot{z})dx - Q'(w, \varphi - \dot{z}) - Q^0(\dot{\psi}, \varphi - \dot{z}), \ \forall \varphi \in S(\Omega)$$

Since

$$\dot{z} = \dot{w} - \dot{\psi} = \dot{w} - \nabla\psi.V \tag{4.23}$$

hence

$$\dot{w} \in S(\Omega) + \nabla\psi.V \ :$$
$$\int_\Omega \Delta\dot{w}\Delta(\varphi - \dot{w})dx \geq \int_\Omega \dot{f}(\varphi - \dot{w})dx - Q'(w, \varphi - \dot{w}) \ , \tag{4.24}$$
$$\forall \varphi \in S(\Omega) + \nabla\psi.V$$

where we denote

$$\dot{\psi} = \nabla\psi.V \in H^2(\Omega), \quad \dot{f} = \ \mathrm{div}(fV),$$
$$Q'(z, \varphi) = \int_\Omega \{-\mathrm{div}V\Delta z\Delta\varphi + \dot{\xi}(z)\Delta\varphi + \Delta z\dot{\xi}(\varphi)\}dx, \quad \forall z, \varphi \in H^2(\Omega),$$

here $\dot{\xi}(\varphi) = \ \mathrm{div}(A'.\nabla\varphi)$, and $A' = \ \mathrm{div}VI - DV - (DV)^*$.

Since the shape derivative depends actually on the normal component $v = \langle V(0), n\rangle_{\mathbb{R}^2}$ of the vector field $V(0,.)$ on $\Gamma = \partial\Omega$, hence for any vector field $V(.,.)$ such that $v(x) = 0$ on $\partial\Omega$ it follows

$$\dot{w} = \nabla w.V(0)$$

and from (4.24) we obtain the following Green formula :

$$0 = -\int_\Omega \Delta(\nabla w.V(0))\Delta\varphi dx + \int_\Omega \dot{f}\varphi dx - Q'(w, \varphi), \quad \forall \varphi \in \{S(\Omega) - S(\Omega)\} \tag{4.25}$$

which holds for any vector field $V(.,.)$ such that $v(x) = 0$ on $\partial\Omega$. For an arbitrary vector field $V(.,.)$ and the test function φ smooth enough, integration by parts in (4.24), in view of (4.13),(4.25), leads to

$$\int_\Omega \Delta w' \Delta(\varphi - w')dx \geq 0 \tag{4.26}$$

futhermore

$$w' \in \{\zeta \mid \zeta = \varphi - \nabla w.V(0), \quad \varphi \in S(\Omega)\} \equiv S_v(\Omega)$$

since we can select $V(0,.)$ with the support in a small open neighborhood of $\partial\Omega$, which completes the proof of Theorem 3.

We refer the reader to [24] for the related results on shape sensitivity analysis of unilateral problems in the Sobolev space $H_0^1(\Omega)$.

4.2. OPTIMAL DESIGN PROBLEM

We derive the necessary optimality conditions for an optimal design problem for the Kirchhoff plate with an obstacle. We refer the reader to [2] where such a problem is defined, and to [16],[24] for the related results on non-smooth optimization problems for the linear elliptic systems.

Let

$$a(h;.,.): \; H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{R} \tag{4.27}$$

be the following bilinear form associated to the Kirchhoff plate [2],[16]

$$a(h;y,\varphi) = \int_\Omega h^3(x) b_{ijkl} \frac{\partial^2 y}{\partial x_i \partial x_j}(x) \frac{\partial^2 \varphi}{\partial x_k \partial x_l}(x)dx, \quad \forall y,\varphi \in H_0^2(\Omega) \tag{4.28}$$

here we use the summation convention over the repeated indices $i,j,k,l = 1,2$. We assume that

$$h \in U_{ad} = \{h \in L^\infty(\Omega) \mid 0 < h_{min} \leq h(x) \leq h_{max}, \quad \text{for a.e. } x \in \Omega\} \tag{4.29}$$

and that the constants $b_{ijkl}, \; i,j,k,l = 1,2$ satisfy the following conditions

$$b_{ijkl} = b_{jikl} = b_{klij}, \quad i,j,k,l = 1,2 \tag{4.30}$$

$$b_{ijkl}\xi_{ij}\xi_{kl} \geq c\xi_{ij}\xi_{ij}, \quad c > 0, \tag{4.31}$$

for all symmetric matrices $[\xi_{ij}]_{2\times 2}$.

We will consider a boundary value problem with the homogenous boundary conditions. However, there is no additional difficulty to derive the same results for the problem with non-homogenous boudary conditions.

It follows by our asumptions (4.29) - (4.31) that the bilinear form (4.28) is continuous, symmetric, and $H_0^2(\Omega)$ - elliptic, i.e.,

$$a(h;y,y) \geq \alpha \|y\|^2_{H^2(\Omega)}, \quad \alpha > 0, \quad \forall y \in H_0^2(\Omega)\,. \tag{4.32}$$

Now let us denote

$$K = \{\varphi \in H_0^2(\Omega) \mid \varphi(x) \geq \psi(x) \text{ in } \Omega\} \tag{4.33}$$

where $\psi(.) \in H^2(\Omega) \subset C(\bar{\Omega})$ is a given element such that the set (4.33) is non-empty, in particular $\psi(x) \leq 0$ on $\Gamma = \partial\Omega$. For a given element $h \in U_{ad}$ we denote by $w = w(h;x)$, $x \in \Omega$, the unique solution of the following variational inequality

$$w \in K \ : \quad a(h; w, \varphi - w) \geq \int_\Omega f(\varphi - w)dx, \quad \forall\varphi \in K \tag{4.34}$$

where $f \in H^{-2}(\Omega)$ is given element, $H^{-2}(\Omega)$ being the dual of $H_0^2(\Omega)$.

Using a variant of Theorem 1 combined with an abstract result of [23] (Theorem 1, p.1419) we obtain the form of the right-differential of the nonlinear mapping

$$L^\infty(\Omega) \ni h \to w(h;.) \in H_0^2(\Omega) \tag{4.35}$$

at a given point $h \in U_{ad}$. This is given in

LEMMA 1

Let μ be the Radon measure defined by

$$\int \varphi d\mu = a(h; w, \varphi) - \int_\Omega f\varphi dx, \quad \forall\varphi \in H_0^2(\Omega)$$

and assume that the support of μ is admissible in the sense of Definition 1.

Then for $\varepsilon > 0, \varepsilon$ small enough

$$\forall v \in L^\infty(\Omega): \quad w(h + \varepsilon v) = w(h) + \varepsilon q(v) + o(\varepsilon) \tag{4.36}$$

where $\|o(\varepsilon)\|_{H_0^2(\Omega)}/\varepsilon \to /0$ with $\varepsilon \downarrow 0$ and $q = q(v) \in H_0^2(\Omega), v \in L^\infty(\Omega)$ is given by the unique solution of the following variational inequality

$$q \in S \ : \quad a(h; q, \varphi - q) + a'_v(h; w(h), \varphi - q) \geq 0, \ \forall\varphi \in S \tag{4.37}$$

where

$$a'_v(h; y, \varphi) = \int_\Omega 3h^2(x)v(x)b_{ijkl}\frac{\partial^2 y}{\partial x_i \partial x_j}(x)\frac{\partial^\varphi}{\partial x_k \partial x_l}(x)dx, \quad \forall y, \varphi \in H_0^2(\Omega) \tag{4.38}$$

$$S = \{\varphi \in H_0^2(\Omega) \mid \varphi = 0 \text{ on spt}\mu, \ \varphi \geq 0 \text{ on } \Xi \setminus \text{spt}\mu\} \tag{4.39}$$

$$\Xi = \{x \in \Omega \mid w(h;x) = \psi(x)\} \quad \text{is compact.} \tag{4.40}$$

Let us consider the following optimal design problem for the Kirchhoff plate.

PROBLEM (P):

Find an element $h \in U_{ad}$ which minimizes the functional

$$J(h) = \max_{x\in\Omega} |w(h;x)|$$

over the set U_{ad}.

We cannot assert in general the existence of an optimal solution $h \in U_{ad}$. This leads to the notion of a generalized solution of problem (P) [16]. We derive the necessary optimality conditions for problem (P) assuming that there exists an optimal solution. The necessary optimality conditions of the same type can be obtained for a generalized solution of problem (P).

THEOREM 4

An optimal solution $h \in U_{ad}$ of problem (P) satisfies the following condition

$$\max_{x \in \Omega^*(h^*)} \text{ sign } \{w(h^*; x)\} q(v - h^*; x) \geq 0, \quad \forall v \in U_{ad} \tag{4.41}$$

where

$$\Omega^*(h) = \{x \in \Omega \mid J(h) = w(h; x)\}, \quad \forall h \in U_{ad} \,.$$

The proof of Theorem 4, in view of Lemma 1, follows by Theorem 1 in [16] and therefore is omitted here.

REMARK 2

An optimal design problem for the Kirchhoff plate with a finite number of pointwise obstacles is considered in [3]. The results derived in [3] are not comparable with our result presented here, since we assume that an obstacle is smooth i.e., $\psi(.) \in H^2(\Omega)$.

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